# INVARIANTS & CATEGORIFICATION
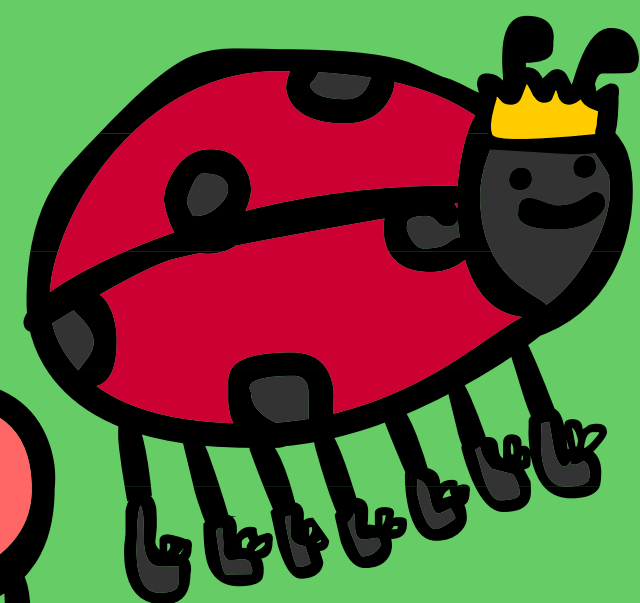
by Tom Hockenhull and Marco Marengon

## INVARIANTS

A good way to tell the difference between two objects is to find a number (like height) or easier to handle thing (like fingerprints) that depends only on the object and helps us tell the difference between things. We call such a thing an 'invariant', and say it is 'perfect' if having different invariants means you are different objects.

WEAR SIZE 9

WEAR SIZE 8

So you can tell them apart by shoe size!

## SHOE SIZE

We can tell two people apart if they wear different sizes of shoe, so it's an invariant. It's not perfect, though!

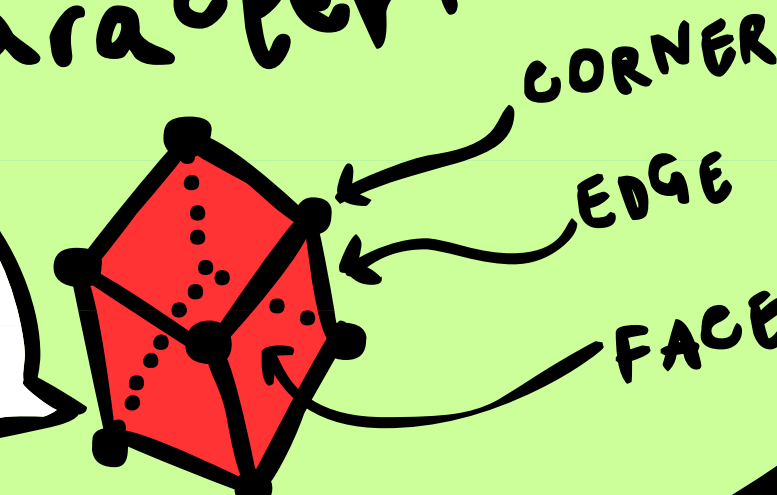To whom does this glass slipper belong?

These pairs of shoes are both size 4. Do they belong to different frogs??

## FEET SIZE

Instead of shoe size, we could try to tell frogs apart by the sizes of their individual feet! This distinguishes between more frogs and its 'average' is shoe size!

I need lots of shoes

WE BOTH WEAR 8

7 8   8 8

## EULER CHARACTERISTIC

For polyhedra, an invariant is the number of corners, minus the number of edges, plus the number of faces. We call it the 'Euler characteristic', $\chi$!
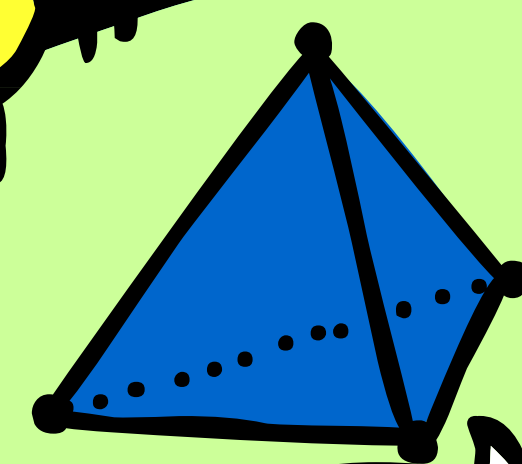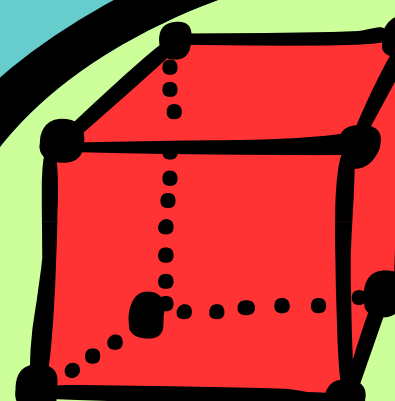
I am cube

CORNER
EDGE
FACE

me $\chi = 2$

me have $\chi = 0$

We can tell them apart with $\chi$!

US have $\chi = 2$
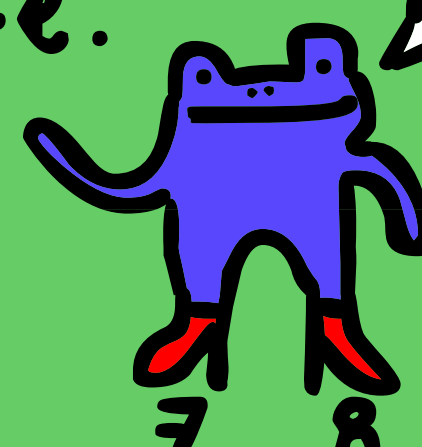
## HOMOLOGY

Instead of $\chi$, we could look at a polyhedron's 'homology': this is a series of objects ('groups'), the 'average' of whose sizes (taken by adding the size of the first to minus the size of the second plus the size of the third, minus the size of the fourth, and so on) is the Euler characteristic.

$Z \Delta \to 0 Z \Delta 0$   ← homology of cube

sizes: 1   0   1   0

and look: $1 - 0 + 1 - 0 = 2 = \chi$ !!!

## ALEXANDER POLYNOMIAL

For a loop of knotted string, an invariant is its 'Alexander-Conway polynomial', written $\Delta$. All you need to know about it is
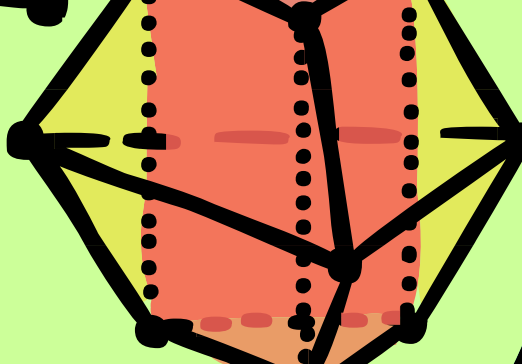
$$z \Delta(\times) = \Delta(\times) - \Delta(\times) \text{ and } \Delta(\bigcirc) = 1,$$

where $z = t^{\frac{1}{2}} - t^{\frac{1}{2}}$. For example:

$$\Delta(\text{knot}) = z \Delta(\text{knot}) + \Delta(\text{knot})$$
$$= z^2 \Delta(\text{knot}) + z \Delta(\text{knot}) + \Delta(\bigcirc)$$
$$= z^2 \cdot 1 + z \cdot 0 + 1$$
$$= (t^{\frac{1}{2}} - t^{\frac{1}{2}})^2 + 1$$

My name is 1134

I'll call you Ian

CLUB $\Delta = 1$

My $\Delta$ is 1

Mine is $t - 1 + t^{-1}$

So they are different!
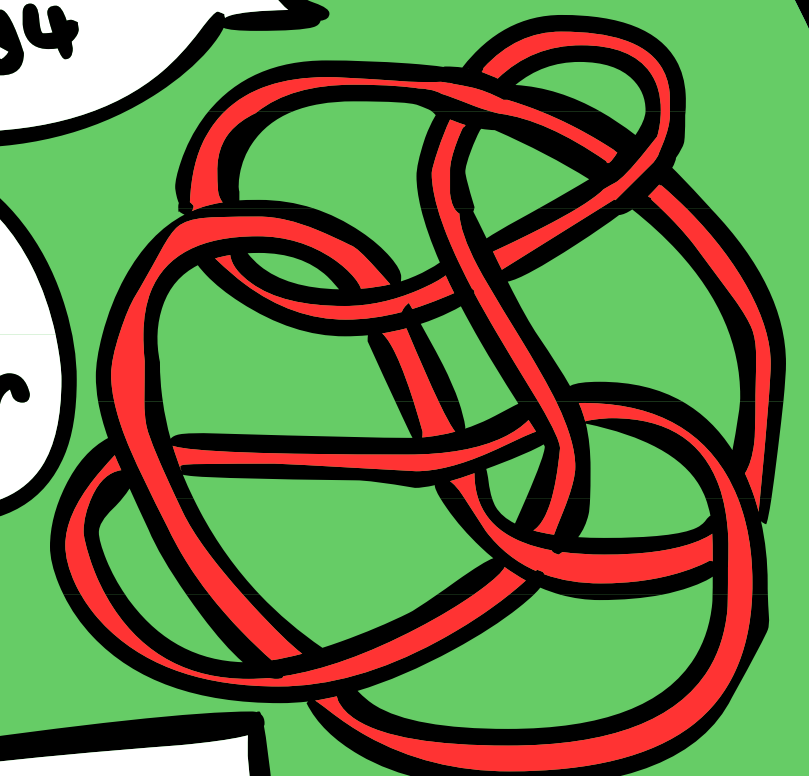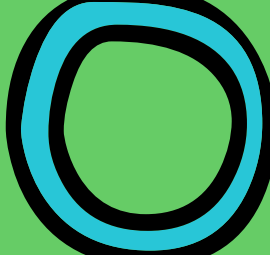
## KNOT FLOER HOMOLOGY

In 2004 Ozsváth, Szabó and Rasmussen discovered (invented?) an invariant called HFK whose 'average' is $\Delta$. It has been shown to distinguish the unknot from everything else by Ozsváth and Szabó.

I am special!

INVARIANTS → FROGETAG → CATEGORIFICATION → KNOTIFICATION → CATEGORIFICATION